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Fourth edition

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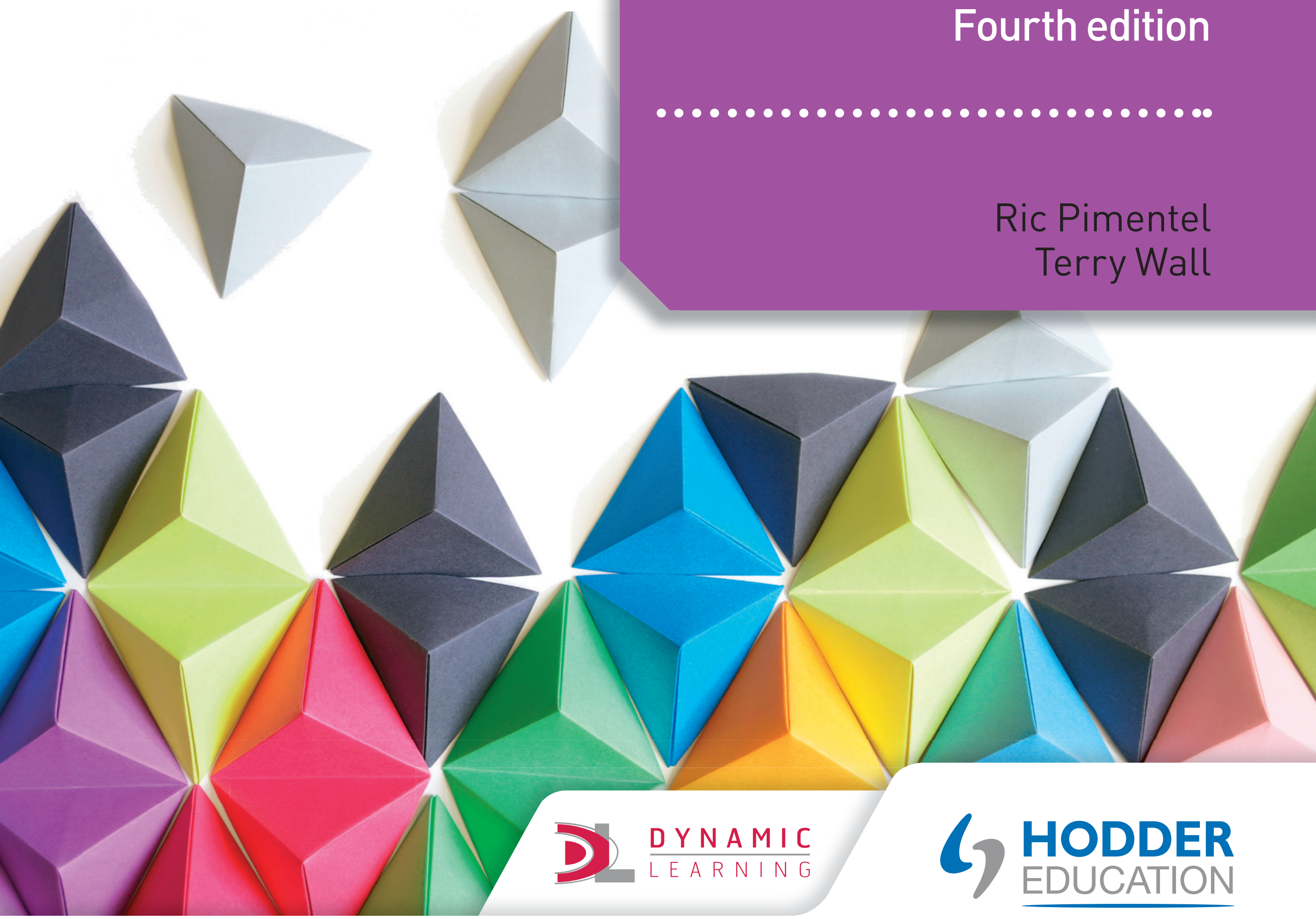

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TOPIC 2

Algebra and graphs

Contents

Syllabus

E2.1

E2.1 Use letters to express generalised numbers and express basic arithmetic processes algebraically. Substitute numbers for words and letters in complicated formulae. Construct and transform complicated formulae and equations.

E2.2

Manipulate directed numbers. Use brackets and extract common factors. Expand products of algebraic expressions. Factorise where possible expressions of the form:

$ax + bx + kay + kby$, $a^2x^2 - b^2y^2$, $a^2 + 2ab + b^2$, $ax^2 + bx + c$

E2.3

Manipulate algebraic fractions. Factorise and simplify rational expressions.

E2.4

Use and interpret positive, negative and zero indices. Use and interpret fractional indices. Use the rules of indices.

E2.5

Derive and solve simple linear equations in one unknown. Derive and solve simultaneous linear equations in two unknowns. Derive and solve quadratic equations by factorisation, completing the square or by use of the formula. Derive and solve simultaneous equations, involving one linear and one quadratic. Derive and solve linear inequalities.

E2.6

Represent inequalities graphically and use this representation to solve simple linear programming problems.

E2.7

Continue a given number sequence. Recognise patterns in sequences including the term to term rule and relationships between different sequences. Find the *n*th term of sequences.

E2.8

Express direct and inverse proportion in algebraic terms and use this form of expression to find unknown quantities.

E2.9

Use function notation, e.g. $f(x) = 3x - 5$, $f: x \to 3x - 5$, to describe simple functions.

Find inverse functions $f^{-1}(x)$.

Form composite functions as defined by $gf(x) = g(f(x))$.

E2.10

Interpret and use graphs in practical situations, including travel graphs and conversion graphs. Draw graphs from given data. Apply the idea of rate of change to simple kinematics involving distance–time and speed–time graphs, acceleration and deceleration. Calculate distance travelled as area under a linear speed–time graph.

E2.11

Construct tables of values and draw graphs for functions of the form ax^n (and simple sums of these) and functions of the form bx. Solve associated equations approximately, including finding and interpreting roots by graphical methods. Draw and interpret graphs representing exponential growth and decay problems. Recognise, sketch and interpret graphs of functions.

E2.12

Estimate gradients of curves by drawing tangents.

E2.13

Understand the idea of a derived function. Use the derivatives of functions of the form ax^n, and simple sums of not more than three of these. Apply differentiation to gradients and turning points (stationary points). Discriminate between maxima and minima by any method.

The Persians

Abu Ja'far Muḥammad ibn Mūsā al-Khwārizmī is called the 'father of algebra'. He was born in Baghdad in AD790. He wrote the book *Hisab al-jabr w'al-muqabala* in AD830 when Baghdad had the greatest university in the world and the greatest mathematicians studied there. He gave us the word 'algebra' and worked on quadratic equations. He also introduced the decimal system from India.

Muhammad al-Karajī was born in North Africa in what is now Morocco. He lived in the eleventh century and worked on the theory of indices. He also worked on an algebraic method of calculating square and cube roots. He may also have travelled to the University in Granada (then part of the Moorish Empire) where works of his can be found in the university library.

The poet Omar Khayyam is known for his long poem *The Rubáiyát*. He was also a fine mathematician working on the binomial theorem. He introduced the symbol 'shay', which became our 'x'.

19 Differentiation and the gradient function

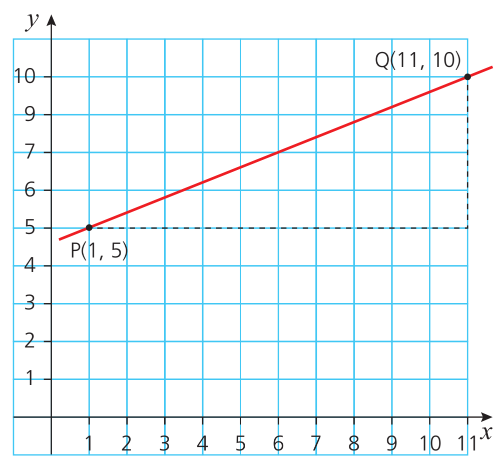
Calculus is the cornerstone of much of the mathematics studied at a higher level. Differential calculus deals with finding the gradient of a function. In this chapter, you will look at functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$, where n is an integer.

The gradient of a straight line

You will already be familiar with calculating the gradient of a straight line. The gradient of the line passing through points (x_1, y_1) and (x_2, y_2) is calculated by $\frac{y_2 - y_1}{x_2 - x_1}$.

Therefore, the gradient of the line passing through points P and Q is:

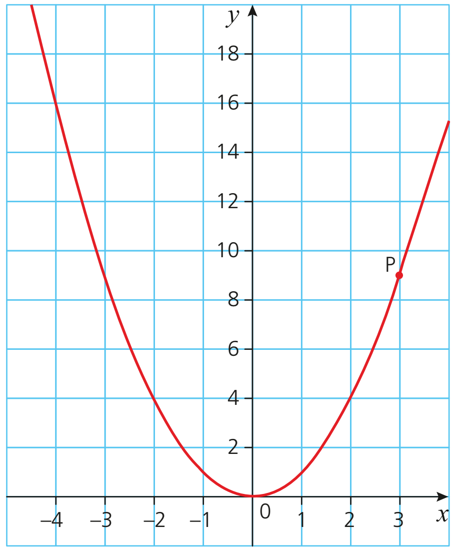
$$\frac{10 - 5}{11 - 1} = \frac{5}{10}$$
$$= \frac{1}{2}$$



The gradient of a curve

The gradient of a straight line is constant, i.e. it is the same at any point on the line. However, not all functions are linear (straight lines). A function that produces a curved graph is more difficult to work with because the gradient of a curve is not constant.

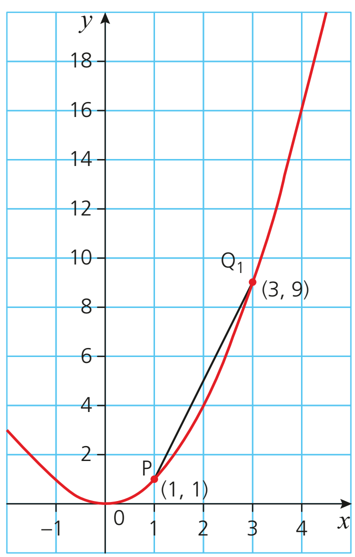
The graph below shows the function $f(x) = x^2$.



Point P is on the curve at (3, 9). If P moves along the curve to the right, the gradient of the curve becomes steeper. If P moves along the curve towards the origin, the gradient of the curve becomes less steep.

The gradient of the function at the point P(1 , 1) can be calculated as follows.

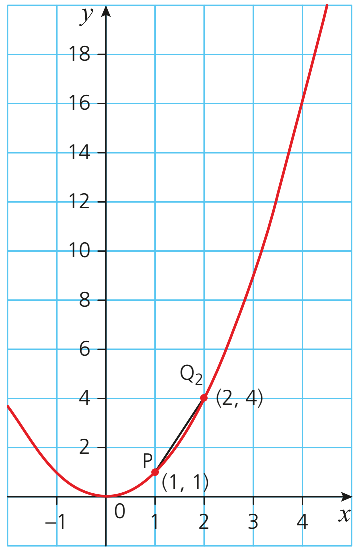
- Mark a point $Q_1(3, 9)$ on the graph and draw the line segment PQ_1.



The gradient of the line segment PQ_1 is an approximation of the gradient of the curve at P.

Gradient of PQ_1 is $\frac{9-1}{3-1} = 4$

» Mark a point Q_2 closer to P, e.g. (2, 4), and draw the line segment PQ_2.



The gradient of the line segment PQ_2 is still only an approximation of the gradient of the curve at P, but it is a better approximation than the gradient of PQ_1.

Gradient of PQ_2 is $\frac{4-1}{2-1} = 3$

» If a point $Q_3(1.5, 1.5^2)$ is chosen, the gradient PQ_3 will be an even better approximation.

Gradient of PQ_3 is $\frac{1.5^2-1}{1.5-1} = 2.5$

For the point $Q_4(1.25, 1.25^2)$, the gradient of PQ_4 is

$\frac{1.25^2-1}{1.25-1} = 2.25$

For the point $Q_5(1.1, 1.1^2)$, the gradient of PQ_5 is $\frac{1.1^2-1}{1.1-1} = 2.1$

These results indicate that as point Q gets closer to P, the gradient of the line segment PQ gets closer to 2.

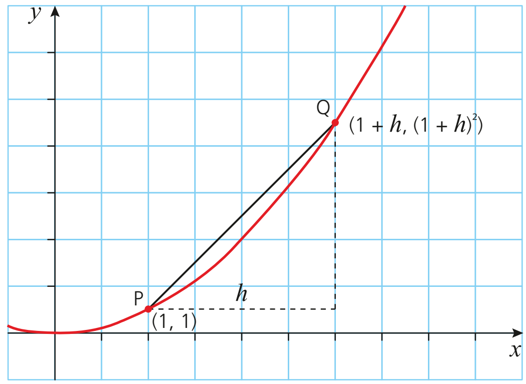
Worked example

Prove that the gradient of the function $f(x) = x^2$ is 2 when $x = 1$.

Solution

Consider points P and Q on the function $f(x) = x^2$.

P is at (1, 1) and Q, h units from P in the x-direction, has coordinates $(1 + h, (1 + h)^2)$.



Gradient of line segment PQ is $\frac{(1+h)^2-1}{1+h-1} = \frac{1+2h+h^2-1}{h}$

$= \frac{h(2+h)}{h}$

$= 2+h$

As Q gets closer to P, h gets smaller and smaller (tends to 0), and the value of $2+h$ becomes an even more accurate approximation of the gradient of the curve at point P.

As h tends to 0, the gradient $(2+h)$ of the line segment PQ tends to 2. This can be written as:

The gradient at P(1, 1) = $\lim_{h\to 0}(2+h) = 2$

In other words, the limit of $2+h$ as h tends to 0 is 2.

In general, **the gradient of a curve at the point P is the same as the gradient of the tangent to the curve at P.**

Exercise 19.1

1 a Using the proof above as a guide, find the gradient of the function $f(x) = x^2$ when:

i) $x = 2$ ii) $x = 3$ iii) $x = -1$.

b Make a table of the values of x and the corresponding values of the gradient of the function $f(x)$.

c Looking at the pattern in your results, complete the sentence below.

For the function $f(x) = x^2$, the gradient is ...

2 a Find the gradient of the function $f(x) = 2x^2$ when:

i) $x = 1$ ii) $x = 2$ iii) $x = -2$.

b Looking at the pattern in your results, complete the sentence below.

For the function $f(x) = 2x^2$, the gradient is ...

3 a Find the gradient of the function $f(x) = \frac{1}{2}x^2$ when:

i) $x = 1$ ii) $x = 2$ iii) $x = 3$.

b Looking at the pattern in your results, complete the sentence below.

For the function $f(x) = \frac{1}{2}x^2$, the gradient is ...

The gradient function

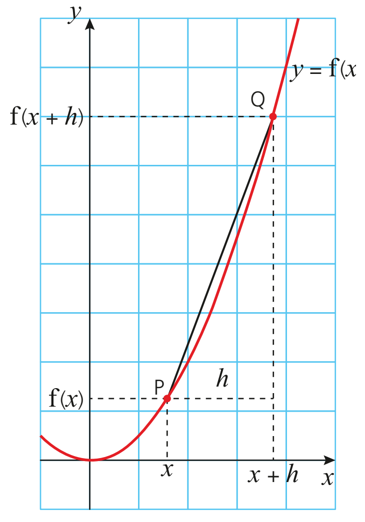
You may have noticed a pattern in your answers to the previous exercise. In fact, there is a rule for calculating the gradient at any point on the particular curve. This rule is known as the **gradient function**, $f'(x)$ or $\frac{dy}{dx}$.

The function $f(x) = x^2$ has a gradient function

$$f'(x) = 2x$$

or $$\frac{dy}{dx} = 2x$$

The above proof can be generalised for other functions $f(x)$.



Gradient of line segment PQ $= \dfrac{f(x+h) - f(x)}{(x+h) - x}$

Gradient at P $= \lim\limits_{h \to 0}(\text{Gradient of line segment PQ})$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is known as finding the gradient function from **first principles**.

Worked example

Find, from first principles, the gradient function of $f(x) = x^2 + x$.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{((x+h)^2+(x+h))-(x^2+x)}{h} \\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2+x+h-x^2-x}{h} \\
&= \lim_{h\to 0}\frac{2xh+h^2+h}{h} \\
&= \lim_{h\to 0}(2x+h+1) \\
&= 2x+1
\end{aligned}$$

So the gradient at any point (x, y) on the curve $y = x^2 + x$ is given by $2x + 1$.

Exercise 19.2

1 Find, from first principles, the gradient function of each function. Use the worked example above as a guide.

- **a** $f(x) = x^3$
- **b** $f(x) = 3x^2$
- **c** $f(x) = x^2 + 2x$
- **d** $f(x) = x^2 - 2$
- **e** $f(x) = 3x - 3$
- **f** $f(x) = 2x^2 - x + 1$

2 Copy and complete the table below using your gradient functions from the previous question and Exercise 19.1 Q1–3.

Function $f(x)$	Gradient function $f'(x)$
x^2	
$2x^2$	
$\frac{1}{2}x^2$	
$x^2 + x$	$2x + 1$

Function f(x)	Gradient function f′(x)
x^3	
$3x^2$	
$x^2 + 2x$	
$x^2 - 2$	
$3x - 3$	
$2x^2 - x + 1$	

3 Look at your completed table for Q2. Describe any patterns you notice between a function and its gradient function.

The functions used so far have all been **polynomials**. There is a relationship between a polynomial function and its gradient function. This is best summarised as follows:

$$\textbf{If } \mathbf{f}(x) = ax^n, \textbf{ then } \frac{dy}{dx} = anx^{n-1}$$

So, to work out the gradient function of a polynomial, multiply the coefficient of x by the power of x and subtract 1 from the power.

Worked examples

1 Calculate the gradient function of $f(x) = 2x^3$.

Solution

$$f'(x) = 3 \times 2x^{(3-1)} = 6x^2$$

2 Calculate the gradient function of $y = 5x^4$.

Solution

$$\frac{dy}{dx} = 4 \times 5x^{(4-1)} = 20x^3$$

Exercise 19.3

1 Calculate the gradient function of each of the following functions:

a $f(x) = x^4$ b $f(x) = x^5$ c $f(x) = 3x^2$

d $f(x) = 5x^3$ e $f(x) = 6x^3$ f $f(x) = 8x^7$

2 Calculate the gradient function of each of the following functions:

a $f(x) = \frac{1}{3}x^3$ b $f(x) = \frac{1}{4}x^4$ c $f(x) = \frac{1}{4}x^2$

d $f(x) = \frac{1}{2}x^4$ e $f(x) = \frac{2}{5}x^3$ f $f(x) = \frac{2}{9}x^3$

Differentiation

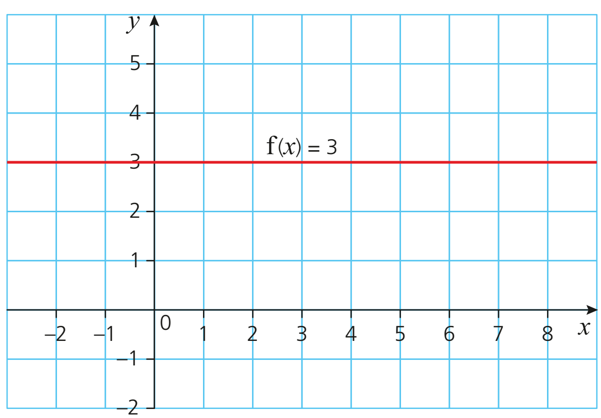
The process of finding the gradient function is known as **differentiation**. Differentiating a function produces the **derivative** or gradient function.

Worked examples

1 Differentiate the function $f(x) = 3$ with respect to x.

Solution

The graph of $f(x) = 3$ is a horizontal line as shown:



A horizontal line has no gradient. Therefore

$$\Rightarrow \text{for } f(x) = 3, \ \frac{dy}{dx} = 0$$

This can also be calculated using the rule for differentiation.

$f(x) = 3$ can be written as $f(x) = 3x^0$.

$$\text{So } \frac{dy}{dx} = 0 \times 3x^{(0-1)}$$
$$= 0$$

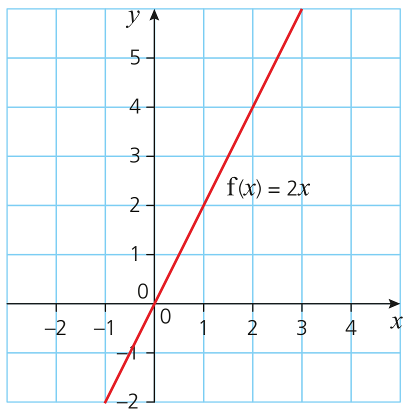
In general, the derivative of a constant is zero.

$$f(x) = c \Rightarrow \frac{dy}{dx} = 0$$

2 Differentiate the function $f(x) = 2x$ with respect to x.

Solution

The graph of $f(x) = 2x$ is a straight line as shown:



From earlier work on linear graphs, the gradient is known to be 2. Therefore

$$\Rightarrow \text{for } f(x) = 2x, \frac{dy}{dx} = 2$$

This too can be calculated using the rule for differentiation.

$f(x) = 2x$ can be written as $f(x) = 2x^1$.

$$\text{So } \frac{dy}{dx} = 1 \times 2x^{(1-1)}$$
$$= 2x^0$$

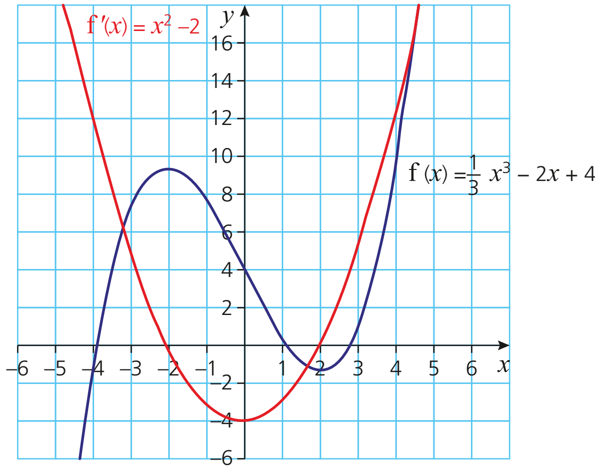
But $x^0 = 1$, therefore $\frac{dy}{dx} = 2$.

In general, $f(x) = ax \Rightarrow \frac{dy}{dx} = a$.

3 Differentiate the function $f(x) = \frac{1}{3}x^3 - 2x + 4$ with respect to x.

Solution

The graphs of the function and its derivative are as follows:



The derivative of the function $f(x)$ is a quadratic. The equation of this quadratic is $y = x^2 - 2$. The derivative of $f(x)$ is therefore $f'(x) = x^2 - 2$.

In general, the derivative of a function with several terms can be found by differentiating each of the terms individually.

4 Differentiate the function $f(x) = \frac{2x^3 + x^2}{x}$ with respect to x.

Solution

A common error here is to differentiate each of the terms individually.

The derivative of $\frac{2x^3 + x^2}{x}$ is NOT $\frac{6x^2 + 2x}{1}$.

$\frac{2x^3 + x^2}{x}$ can be written as $\frac{2x^3}{x} + \frac{x^2}{x}$ and simplified to $2x^2 + x$.

Therefore
$$f(x) = \frac{2x^3 + x^2}{x}$$
$$= 2x^2 + x$$
$$\Rightarrow \frac{dy}{dx} = 4x + 1$$

In general, rewrite functions as sums of terms in powers of x before differentiating.

Exercise 19.4

1 Differentiate each expression with respect to x.

a $5x^3$ **b** $7x^2$ **c** $4x^6$

d $\frac{1}{4}x^2$ **e** $\frac{2}{3}x^6$ **f** $\frac{3}{4}x^5$

g 5 **h** $6x$ **i** $\frac{1}{8}$

2 Differentiate each expression with respect to x.

a $3x^2 + 4x$ **b** $5x^3 - 2x^2$

c $10x^3 - \frac{1}{2}x^2$ **d** $6x^3 - 3x^2 + x$

e $12x^4 - 2x^2 + 5$ **f** $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 4$

g $-3x^4 + 4x^2 - 1$ **h** $-6x^5 + 3x^4 - x + 1$

i $-\frac{3}{4}x^6 + \frac{2}{3}x^3 - 8$

3 Differentiate each expression with respect to x.

a $\frac{x^3 + x^2}{x}$ **b** $\frac{4x^3 - x^2}{x^2}$

c $\frac{6x^3 + 2x^2}{2x}$ **d** $\frac{x^3 + 2x^2}{4x}$

e $3x(x + 1)$ **f** $2x^2(x - 2)$

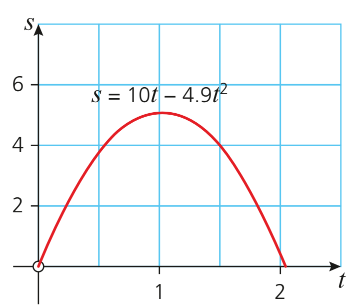
g $(x + 5)^2$ **h** $(2x - 1)(x + 4)$

i $(x^2 + x)(x - 3)$

So far, we have only used the variables x and y when finding the gradient function. This does not always need to be the case. Sometimes, as demonstrated below, it is more convenient or appropriate to use other variables.

If a stone is thrown vertically upwards from the ground with a speed of $10\ ms^{-1}$, its distance (s) from its point of release is given by the formula $s = 10t - 4.9t^2$, where t is the time in seconds after the stone's release.

A graph showing distance against time is plotted:



The velocity (v) of the stone at any point can be found by calculating the rate of change of distance with respect to time, i.e. $\frac{ds}{dt}$.

Therefore if $s = 10t - 4.9t^2$

$$v = \frac{ds}{dt}$$
$$= 10 - 9.8t$$

Worked example

Calculate $\frac{ds}{dt}$ for the function $s = 6t^2 - 4t + 1$.

Solution

$$\frac{ds}{dt} = 12t - 4$$

Exercise 19.5

1 Differentiate each of the following with respect to t.

- a $y = 3t^2 + t$
- b $v = 2t^3 - t^2$
- c $m = 5t^3 - t^2$

2 Calculate the derivative of each of the following functions.

- a $y = x(x + 4)$
- b $r = t(1 - t)$
- c $v = t\left(\frac{1}{t} + t^2\right)$
- d $p = r^2\left(\frac{2}{r} - 3\right)$

3 Differentiate each of the following with respect to t.

- a $y = (t + 1)(t - 1)$
- b $r = (t - 1)(2t + 2)$
- c $v = \left(\frac{2t^2}{3} + 1\right)(t - 1)$

Calculating the second derivative

In the previous section we considered at the position of a stone thrown vertically upwards. Its velocity (v) at any point was found by differentiating the equation for the distance (s) with respect to t, i.e. $v = \frac{ds}{dt}$.

In this section, we extend this to consider acceleration (a) which is the rate of change of velocity with time, i.e. $a = \frac{dv}{dt}$.

Therefore, as $s = 10t - 4.9t^2$

$$v = \frac{ds}{dt}$$
$$= 10 - 9.8t$$
$$a = \frac{dv}{dt}$$
$$= -9.8$$

acceleration due to gravity

You will have noticed that the equation for the distance was differentiated twice to get the acceleration. i.e. the second derivative was obtained. Calculating the second derivative is a useful operation as will be seen later.

The notation used for the second derivative follows on from that used for the first derivative.

$$\mathrm{f}(x) = ax^n \qquad\qquad y = ax^n$$
$$\Rightarrow \mathrm{f}'(x) = anx^{n-1} \qquad \text{or} \qquad \frac{dy}{dx} = anx^{n-1}$$
$$\Rightarrow \mathrm{f}''(x) = an(n-1)x^{n-2} \qquad\qquad \frac{d^2y}{dx^2} = an(n-1)x^{n-2}$$

Therefore either $\mathrm{f}''(x)$ or $\frac{d^2y}{dx^2}$ are the most common form of notation used for the second derivative, when differentiating with respect to x.

Worked examples

1 Find $\frac{d^2y}{dx^2}$ when $y = x^3 - 2x^2$.

Solution

$$\frac{dy}{dx} = 3x^2 - 4x$$
$$\frac{d^2y}{dx^2} = 6x - 4$$

2 Work out $\frac{d^2s}{dt^2}$ for $s = 3t + \frac{1}{2}t^2$.

Solution

$$\frac{ds}{dt} = 3 + t$$
$$\frac{d^2s}{dt^2} = 1$$

Exercise 19.6

1 Find $\dfrac{d^2y}{dx^2}$ for each of the following:

a $y = 2x^3$ **b** $y = x^4 - \frac{1}{2}x^2$ **c** $y = \frac{1}{3}x^6$

d $y = 3x^2 - 2$ **e** $y = \dfrac{x^2}{4}$ **f** $y = 3x$

2 Find the second derivative of each function:

a $v = x^2(x - 3)$ **b** $P = \frac{1}{2}x^2(x^2 + x)$ **c** $t = x^{-1}(x + x^3)$

d $y = (x^2 + 1)(x^3 - x)$

Gradient of a curve at a point

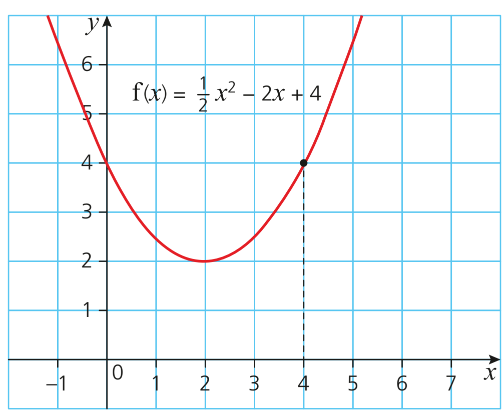
You have seen that differentiating the equation of a curve gives the general equation for the gradient of any point on the curve. You can use this general equation to calculate the gradient at a specific point on the curve.

For the function $f(x) = \frac{1}{2}x^2 - 2x + 4$, the gradient function $f'(x) = x - 2$.

The gradient at any point on the curve can be calculated using this.

For example, when $x = 4$, $f'(4) = 4 - 2$
$= 2$

Therefore, the gradient of the curve $f(x) = \frac{1}{2}x^2 - 2x + 4$ is 2 when $x = 4$, as shown below.



The gradient of the curve at $x = 4$ is 2.

Worked example

Calculate the gradient of the curve $f(x) = x^3 + x - 6$ when $x = -1$.

Solution

The gradient function $f'(x) = 3x^2 + 1$.

$$\text{So, when } x = -1,\ f'(-1) = 3(-1)^2 + 1 = 4$$

Therefore the gradient is 4.

Exercise 19.7

1 Find the gradient of each function at the given value of x.

a $f(x) = x^2$, $x = 3$

b $f(x) = \frac{1}{2}x^2 - 2$, $x = -3$

c $f(x) = 3x^3 - 4x^2 - 2$, $x = 0$

d $f(x) = -x^2 + 2x - 1$, $x = 1$

e $f(x) = -\frac{1}{2}x^3 + x - 3$, $x = -1$, $x = 2$

f $f(x) = 6x$, $x = 5$

2 The number of newly-infected people, N, on day t of a stomach bug outbreak is given by $N = 5t^2 - \frac{1}{2}t^3$.

a Calculate the number of new infections N when:
i) $t = 1$ ii) $t = 3$ iii) $t = 6$ iv) $t = 10$.

b Calculate the rate of new infections with respect to t, i.e. calculate $\frac{dN}{dt}$.

c Calculate the rate of new infections at the following times:
i) $t = 1$ ii) $t = 3$ iii) $t = 6$ iv) $t = 10$.

d Plot a graph of the equation $N = 5t^2 - \frac{1}{2}t^3$ for the values of t in the range $0 \le t \le 10$.

e Explain your answers to part **a**, using your graph to support your explanation.

f Explain your answers to part **c**, using your graph to support your explanation.

4 A weather balloon is released from the ground. Its height in metres, h, after time in hours, t, is given by the formula:

$h = 30t^2 - t^3$

when $t \le 20$.

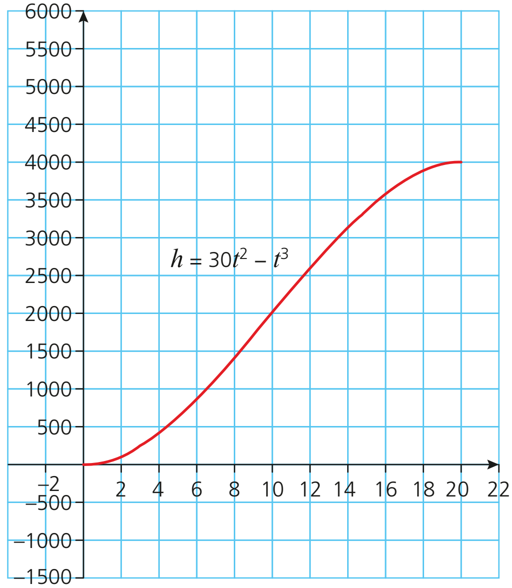
a Calculate the height of the balloon when:
i) $t = 3$ ii) $t = 10$.

b Calculate the rate at which the balloon is climbing with respect to time t.

c Calculate the rate of ascent when:

i) $t = 2$ **ii)** $t = 5$ **iii)** $t = 20$.

d The graph of h against t is shown below.



Referring to your graph, explain your answers to part 'c'

e Estimate from your graph the time when the balloon was climbing at its fastest rate. Explain your answer

Calculating the value of x when the gradient is given

So far, you have calculated the gradient of a curve for a given value of x. It is also possible to work backwards and calculate the value of x when the gradient of a point is given.

Consider the function $f(x) = x^2 - 2x + 1$.

It is known that the gradient at a particular point on the curve is 4, but the x-coordinate of that point is not known.

The gradient function of the curve is $f'(x) = 2x - 2$.

Since the gradient at this particular point is 4, you can form an equation:

$$f'(x) = 4$$

So $2x - 2 = 4$

$$\Rightarrow 2x = 6$$

$$\Rightarrow x = 3$$

Therefore $x = 3$ when the gradient of the curve is 4.

Worked example

The function $f(x) = x^3 - x^2 - 5$ has a gradient of 8 at a point P on the curve.

Calculate the possible coordinates of point P.

Solution

The gradient function $f'(x) = 3x^2 - 2x$

At P, $3x^2 - 2x = 8$

This can be rearranged into the quadratic $3x^2 - 2x - 8 = 0$ and solved algebraically.

This requires the algebraic solution of the quadratic equation $3x^2 - 2x - 8 = 0$

Factorising gives $(3x + 4)(x - 2) = 0$

Therefore $(3x + 4) = 0 \Rightarrow x = -\frac{4}{3}$ or $(x - 2) = 0 \Rightarrow x = 2$

The values of $f(x)$ can be calculated by substituting the x-values in to the equation as shown:

$$f\left(-\frac{4}{3}\right) = \left(-\frac{4}{3}\right)^3 - \left(-\frac{4}{3}\right)^2 - 5 = -9\frac{4}{27}$$

$$f(2) = 2^3 - 2^2 - 5 = -1$$

Therefore the possible coordinates of P are $\left(-1\frac{1}{3}, -9\frac{4}{27}\right)$ and $(2, -1)$.

Exercise 19.8

1 Find the coordinate of the point P on each of the following curves, at the given gradient.

a $f(x) = x^2 - 3$, gradient at P = 6

b $f(x) = 3x^2 + 1$, gradient at P = 15

c $f(x) = 2x^2 - x + 4$, gradient at P = 7

d $f(x) = \frac{1}{2}x^2 - 3x - 1$, gradient at P = −3

e $f(x) = \frac{1}{3}x^2 + 4x$, gradient at P = 6

f $f(x) = -\frac{1}{5}x^2 + 2x + 1$, gradient at P = 4

2 Find the coordinate(s) of the point (s) on each of the following curves, at the given gradient.

a $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x$, gradient = 6

b $f(x) = \frac{1}{3}x^3 + 2x^2 + 6x$, gradient = 3

c $f(x) = \frac{1}{3}x^3 - 2x^2$, gradient = −4

d $f(x) = x^2 - x^2 + 4x$, gradient = 5

3 A stone is thrown vertically downwards off a tall cliff. The distance (s) it travels in metres is given by the formula $s = 4t + 5t^2$, where t is the time in seconds after the stone's release.

a What is the rate of change of distance with time, $\frac{ds}{dt}$? (This represents the velocity.)

b How many seconds after its release is the stone travelling at a velocity of 9 m/s?

c The speed of the stone as it hits the ground is 34 m/s. How many seconds after its release did the stone hit the ground?

d Using your answer to part **c**, calculate the distance the stone falls and hence the height of the cliff.

4 The temperature inside a pressure cooker (T) in degrees Celsius is given by the formula

$$T = 20 + 12t^2 - t^3$$

where t is the time in minutes after the cooking started and $t \leq 8$.

a Calculate the initial (starting) temperature of the pressure cooker.

b What is the rate of temperature increase with time?

c What is the rate of temperature increase when:

i $t = 1$ **ii** $t = 4$ **iii** $t = 8$.

d The pressure cooker was switched off when $\frac{dT}{dt} = 36$.

How long after the start could the pressure cooker have been switched off? Give both possible answers.

e What was the final temperature of the pressure cooker if it was switched off at the greater of the two times calculated in part **d**.

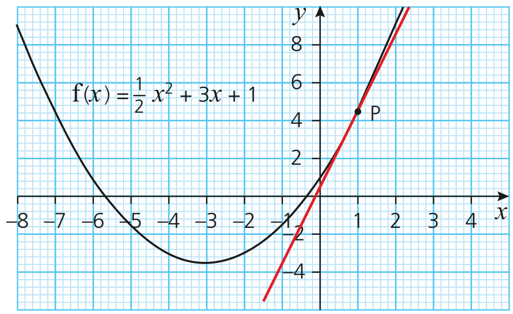
Equation of the tangent at a given point

You already know that the gradient of a tangent drawn at a point on a curve is equal to the gradient of the curve at that point.

Worked example

Find the equation of the tangent of $f(x) = \frac{1}{2}x^2 + 3x + 1$.

The function $f(x) = \frac{1}{2}x^2 + 3x + 1$ has a gradient function of $f'(x) = x + 3$.



Solution

At point P, where $x = 1$, the gradient of the curve is 4.

The tangent drawn to the curve at P also has a gradient of 4.

The equation of the tangent can also be calculated. As it is a straight line, it must take the form $y = mx + c$. The gradient m is 4 as shown above.

Therefore $y = 4x + c$.

As the tangent passes through the point $P\left(1, 4\frac{1}{2}\right)$, these values can be substituted for x and y so that c can be calculated.

$$4\frac{1}{2} = 4 + c$$

$$\Rightarrow c = \frac{1}{2}$$

The equation of the tangent is therefore $y = 4x + \frac{1}{2}$.

Exercise 19.9

1 For the function $f(x) = x^2 - 3x + 1$

a Calculate the gradient function.

b Calculate the gradient of the curve at the point A(2, 1).
A tangent is drawn to the curve at A.

c What is the gradient of the tangent?

d Calculate the equation of the tangent. Give your answer in the form $y = mx + c$.

2 For the function $f(x) = 2x^2 - 4x - 2$

a Calculate the gradient of the curve at $x = 2$.
A tangent is drawn to the curve at the point (2, –2).

b Calculate the equation of the tangent. Give your answer in the form $y = mx + c$.

3 A tangent is drawn to the curve $f(x) = \frac{1}{2}x^2 - 4x - 2$ at the point P(0, –2).

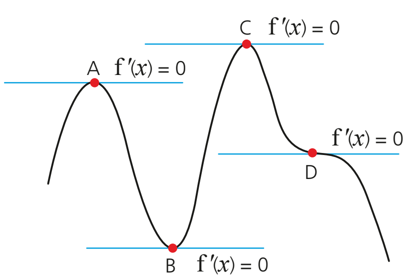
a Calculate the gradient of the tangent at P.

b Calculate the equation of the tangent. Give your answer in the form $y = mx + c$.

4 A tangent, T_1, is drawn to the curve $f(x) = -x^2 + 4x + 1$ at point A(4, 1).
   a Calculate the gradient of the tangent at A.
   b Calculate the equation of the tangent. Give your answer in the form $y = mx + c$.
   c Another tangent to the curve, T_2, is drawn at point B(2, 5). Calculate the equation of T_2.

5 A tangent, T_1, is drawn to the curve $f(x) = -\frac{1}{4}x^2 - 3x + 1$ at point P(−2, 6).
   a Calculate the equation of T_1.
   Another tangent to the curve, T_2, with equation $y = 10$, is drawn at point Q.
   b Calculate the coordinates of point Q.
   c T_1 and T_2 are extended so that they intersect. Calculate the coordinates of their point of intersection.

6 The equation of a tangent T, drawn to the curve $f(x) = -\frac{1}{2}x^2 - x - 4$ at P, has equation $y = -3x - 6$.
   a Calculate the gradient function of the curve.
   b What is the gradient of the tangent T?
   c What are the coordinates of point P?



Stationary points

There are times when the gradient of a point on a curve is zero, i.e. the tangent drawn at that point is horizontal. A point where the gradient of the curve is zero is known as a **stationary point**. There are different types of stationary point.

Points A and C are **local maxima**, point B is a **local minima** and point D is a **point of inflexion**. This text covers local maximum and minimum points only.

As the worked example shows, it is not necessary to sketch a graph in order to find the position of any stationary points or to identify what type of stationary points they are.

Worked example

a A graph has equation $y = \frac{1}{3}x^3 - 4x + 5$. Find the coordinates of the stationary points on the graph.

b Determine the nature of each of the stationary points.

Solution

a If $y = \frac{1}{3}x^3 - 4x + 5$, $\frac{dy}{dx} = x^2 - 4$.

At a stationary point $\frac{dy}{dx} = 0$, so solve $x^2 - 4 = 0$ to find the x-coordinate of any stationary point.

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Substitute $x = 2$ and $x = -2$ into the equation of the curve to find the corresponding y-coordinates.

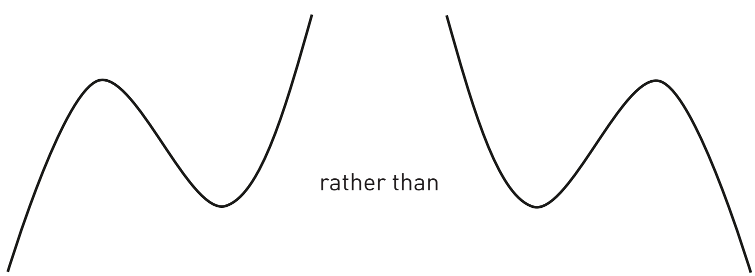
When $x = 2$ $\quad y = \frac{1}{3}(2)^3 - 4(2) + 5 = -\frac{1}{3}$

When $x = -2$ $\quad y = \frac{1}{3}(-2)^3 - 4(-2) + 5 = 10\frac{1}{3}$

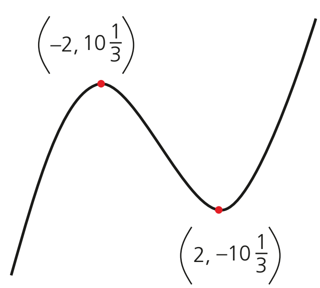
The coordinates of the stationary points are $\left(2, -\frac{1}{3}\right)$ and $\left(-2, 10\frac{1}{3}\right)$.

b There are several methods that can be used to establish the type of stationary point.

Graphical deduction
As the curve is a cubic and the coefficient of the x^3 term is positive, the shape of the curve is of the form



Therefore it can be deduced that the positions of the stationary points are:



Hence $\left(-2, 10\frac{1}{3}\right)$ is a maximum point and $\left(2, -\frac{1}{3}\right)$ a minimum point.

Gradient inspection

The gradient of the curve either side of a stationary point can be calculated.

At the stationary point where $x = 2$, consider the gradient at $x = 1$ and at $x = 3$.

$\frac{dy}{dx} = x^2 - 4,$ so when $x = 1$, $\frac{dy}{dx} = -3$

and when $x = 3$, $\frac{dy}{dx} = 5$

As x increases, the gradient changes from negative to positive, therefore the stationary point must be a minimum.

At the stationary point where $x = -2$, consider the gradient at $x = -3$ and at $x = 1$.

$\frac{dy}{dx} = x^2 - 4,$ so when $x = -3$, $\frac{dy}{dx} = 5$

and when $x = -1$, $\frac{dy}{dx} = -3$

As x increases, the gradient changes from positive to negative, therefore the stationary point must be a maximum.

The second derivative

The second derivative, $\frac{d^2y}{dx^2}$, is usually the most efficient way of determining whether a stationary point is a maximum or minimum.

The proof is beyond the scope of this book, however the general rule is that:

$\frac{d^2y}{dx^2} < 0 \Rightarrow$ a maximum point

$\frac{d^2y}{dx^2} > 0$ a minimum point.

In this example, $\frac{dy}{dx} = x^2 - 4$ and $\frac{d^2y}{dx^2} = 2x$.

Substituting the x -values (2 and −2) of the stationary points into $\frac{d^2y}{dx^2}$ gives:

$\frac{d^2y}{dx^2} = 2(-2) = -4$ (a maximum point)

and $\frac{d^2y}{dx^2} = 2(2) = 4$ (a minimum point).

Note: When $\frac{d^2y}{dx^2} = 0$, the stationary point could either be a maximum or a minimum point, so another method should be used.

Exercise 19.10

1 For each function, calculate:

i) the gradient function
ii) the coordinates of any stationary points.

a $f(x) = x^2 - 6x + 13$
b $f(x) = x^2 + 12x + 35$
c $f(x) = -x^2 + 8x - 13$
d $f(x) = -6x + 7$

2 For each function, calculate:

i) the gradient function
ii) the coordinates of any stationary points.

a $f(x) = x^3 - 12x^2 + 48x - 58$
b $f(x) = x^3 - 12x$
c $f(x) = x^3 - 3x^2 + 45x + 8$
d $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x - 5$

For questions 3–6:
i) Calculate the gradient function.
ii) Calculate the coordinates of any stationary points.
iii) Determine the type of each stationary point.
iv) Calculate the value of the y -intercept.
v) Sketch the graph of the function.

3 $f(x) = 1 - 4x - x^2$

4 $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 3$

5 $f(x) = -\frac{2}{3}x^3 + 3x^2 - 4x$

6 $f(x) = x^3 - \frac{9}{2}x^2 - 30x + 4$

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